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# Heun equations and quasi exact solvability

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**Abstract.** We consider the Heun equations in the context of the quasi-exactly-solvable spectral problems and establish the conditions for this class of equations to admit algebraic solutions. The Schrödinger operators that can be associated with Heun equations are constructed explicitly; in some cases, we present their supersymmetric partners. We study a Heun-like system of equations which is relevant for the classical solution of the two-dimensional Abelian–Higgs model.

## 1. Introduction

In quantum mechanics, there are a few physically meaningful Schrödinger equations (e.g. the harmonic oscillator or the Coulomb problem) whose entire spectra are explicitely calculable. Extracting a suitable prefactor from the wavefunction and using an appropriate variable, the spectral problem can be reduced to a differential equation whose relevant solutions are polynomials. We will refer to these examples as exactly solvable (ES) equations.

Besides these equations, there are simple potentials (e.g. the celebrated quartic potential) for which no one eigenstate can be expressed in terms of elementary functions.

A few years ago, a new class of eigenvalue equations was introduced which occupy an intermediate position between the two extreme cases mentioned above [1–3]. They are the quasi-exactly-solvable (QES) equations. For these equations, a part of the spectrum (generally a finite number of eigenstates) can be computed by algebraic methods. As for ES equations, a suitable change of function and of variable allows one to express the operator defining the QES equation in a form that leaves the space of polynomials of fixed degree, say $n$, invariant. Accordingly, $n+1$ eigenstates can be found by solving the eigenvalue problem for an $(n+1)\times(n+1)$ numerical matrix. This is why one calls the corresponding eigenvectors and eigenvalues 'algebraically computable' or, for shortness, 'algebraic'. Unlike ES equations, the algebraic solutions of QES equations exist for specific values (depending on $n$) of the coupling constant.

The ES equations are typically related to hypergeometric equations which have three regular singularities. For many QES spectral problems, the relevant differential equation has four regular singularities; accordingly, they belong to the class of Heun equations. Therefore, it is worth studying the relationship between the Heun and QES equations; this is the aim of the present paper.

We present a general description of Heun equations in section 2 and, in section 3, we analyse the conditions for such equations to be QES. In section 4, we discuss a

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*meaningful spectral problem which, like Heun equations, admits a rich set of algebraic* solutions. Finally, section 5 approaches the relationship between QES Heun equations and supersymetric quantum mechanics.

## 2. Heun equation

In this section, we present several formulations of the Heun equation.

### 2.1. General definition

The Heun equation [4] can be defined by means of the following Fuchs operator:

$$4t(1-t)(1-k^2t)\frac{d^2G}{dt^2}+2\{(\alpha-1)k^2t(1-t)+(\beta-1)t(1-k^2t)+(\gamma+1)(1-t)(1-k^2t)\}\frac{dG}{dt} + (\lambda-\delta k^2t)G=0 \qquad 0\leqslant k^2\leqslant 1 \tag{1}$$

where $\alpha, \beta, \gamma, \lambda, \delta$ and $k^2$ are constants. It has four regular singularities (canonically chosen at $t = 0, k^2, 1, \infty$); in this respect, it differs from the hypergeometric equation which has three regular singularities. Using the new variable and function

$$t = \mathrm{sn}^2(x,k) \qquad F(x) = G(t) \tag{2}$$

equation (1) can be set in the form

$$H(\alpha,\beta,\gamma,\delta)F(x) = -\lambda F(x) \tag{3}$$

with

$$H(\alpha,\beta,\gamma,\delta) \equiv \frac{d^2}{dx^2}+\left[\frac{\alpha k^2\,\mathrm{sn}^2\,\mathrm{cn}^2+\beta\,\mathrm{sn}^2\,\mathrm{dn}^2+\gamma\,\mathrm{cn}^2\,\mathrm{dn}^2}{\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}}\right]\frac{d}{dx}-\delta k^2\,\mathrm{sn}^2. \tag{4}$$

*Here, sn, cn, dn represent the Jacobi elliptic functions sn(x, k), cn(x, k), dn(x, k) of modulus* $k$; $\mathrm{sn}(x,k)$ and $\mathrm{cn}(x,k)$ are periodic with period $4K(k)$, while $\mathrm{dn}(x,k)$ has a period $2K(k)$; $K(k)$ is the complete elliptic function of the first kind.

### 2.2. Schrödinger form

Changing $F(x)$ in equation (3) into a new function $\psi(x)$, such that

$$\psi(x) = \sqrt{\mathrm{sn}^{\gamma}\,\mathrm{cn}^{-\beta}\,\mathrm{dn}^{-\alpha}}\,F(x) \tag{5}$$

one obtains a Schrödinger equation for $\psi(x)$

$$-\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x) = h\psi(x) \tag{6}$$

with

$$V(x) = \Delta k^2\,\mathrm{sn}^2+\frac{\gamma(\gamma-2)}{4\,\mathrm{sn}^2}+\frac{\beta(\beta+2)(1-k^2)}{4\,\mathrm{cn}^2}+\frac{\alpha(\alpha+2)(k^2-1)}{4\,\mathrm{dn}^2} \tag{7}$$

$$\Delta = \delta+(\alpha+\beta-\gamma-1)^2-1 \tag{8}$$

$$h = \lambda-\tfrac{1}{4}((\alpha+1)^2-(\beta+\gamma)^2-1)-\tfrac{1}{4}k^2((\beta+1)^2-(\alpha+\gamma)^2-1). \tag{9}$$

The potential $V(x)$ is periodic with period $2K(k)$ and, in general, singular (only the function dn is free of zeros on the real axis).

### 2.3. Trigonometric form

Shifting the variable $x$ in equation (3) by one quarter of the period and changing to a new function and variable by means of

$$\tilde{F}(x - K(k)) = \mathrm{dn}^{2\eta}\,\Phi(\phi) \qquad \sin(\phi) = \mathrm{sn}(x) \tag{10}$$

($\eta$ is defined below in equation (13)) enables one to express equations (3) and (4) in a form that depends only on the usual trigonometric functions

$$\tilde{H}\Phi(\phi) = \lambda\Phi(\phi) \tag{11}$$

with

$$\tilde{H} = (1 - k^2\sin^2\phi)\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} - \left[k^2(\alpha - 4\eta + 1)\sin\phi\cos\phi + \beta\frac{\cos\phi}{\sin\phi} + \gamma(1 - k^2)\frac{\sin\phi}{\cos\phi}\right]\frac{\mathrm{d}}{\mathrm{d}\phi}$$
$$+ k^2(4\eta^2 - 2\eta - 2\eta\alpha)\sin^2\phi \qquad \sin\phi = \mathrm{sn}\,x. \tag{12}$$

This form of Heun equation was used in relation with spin systems [7].

## 3. Quasi-exactly-solvable Heun equations

### 3.1. General case

In order to discuss the Heun equations in the framework of the QES equations [1–3], it is convenient to define the parameter $\eta$ as follows

$$\delta = 2\eta(2\eta + 1) + 2\eta(\gamma - \alpha - \beta). \tag{13}$$

If $\eta$ is an integer, say $\eta = n$, then one can show easily that the Fuchs operator in equation (1) preserves $\mathcal{P}_n$, the space of polynomials of degree $n$ in the variable $t$. Therefore, the form (13) of $\delta$ with $\eta = n$ guarantees the existence of $n+1$ polynomial solutions $G(t)$, corresponding to $n + 1$ particular values of parameter $\lambda$. If, on the other hand,

$$m \equiv \frac{\alpha + \beta - \gamma - 1}{2} - n \tag{14}$$

is also a positive integer, then operator (1) preserves both $\mathcal{P}_n$ and $\mathcal{P}_m$ [5]. In these cases, polynomial solutions of degrees $m$ and $n$ exist.

Using equation (2), one can associate, to any polynomial solution of equation (1), an 'algebraic' eigenvector to the spectral problem (3). The eigenvalue equation is considered with a periodic boundary condition, where $\lambda$ is the spectral parameter. The eigenvectors produced in this way have a period $2K(k)$.

The corresponding solutions to equation (6) can also be obtained; due to the square-root factor in equation (5), these solutions do not, in general, fulfil the physical requirements imposed on wavefunctions.

One may ask the following question concerning equation (3): 'given $\alpha, \beta, \gamma$, which values of $\delta$ allow algebraic eigenvectors to exist?' For generic values of $\alpha, \beta, \gamma$, the possible values of $\delta$ assemble into eight families, each indexed by an integer. One obtains this result once the following property of operator (3) is observed [6]. Let $a, b, c$ be three real numbers such that

$$a(a-1-\alpha) = b(b-1-\beta) = c(c-1+\gamma) = 0. \tag{15}$$

Then, we have the following identity for the operators (4):

$$\operatorname{sn}^{-c}\operatorname{cn}^{-b}\operatorname{dn}^{-a} H(\alpha,\beta,\gamma,\eta)\operatorname{sn}^{c}\operatorname{cn}^{b}\operatorname{dn}^{a} = H\left(\alpha-2a, \beta-2b, \gamma+2c, \eta-\frac{a+b+c}{2}\right) + c(1-\beta) + b(1+\gamma) + 2bc + k^2(a(1+\gamma) + c(1-\alpha) + 2ac). \tag{16}$$

One can therefore construct algebraic solutions of the form

$$F(x) = \operatorname{sn}^{c}\operatorname{cn}^{b}\operatorname{dn}^{a} P_n(\operatorname{sn}^2) \qquad n = \eta - \frac{a+b+c}{2} \qquad P_n \in \mathcal{P}_n \tag{17}$$

if $a, b, c$ obey equation (15) and $n$ is a positive integer. One then finds easily the eight families of values of $\delta$:

$$\begin{array}{lll}
(a) & \delta = 2n(2n+1+\gamma-\alpha-\beta) & \text{if } a=0, b=0, c=0 \\
(b) & \delta = (2n+1-\gamma)(2n+2-\alpha-\beta) & \text{if } a=0, b=0, c=1-\gamma \\
(c) & \delta = (2n+1+\beta)(2n+2+\gamma-\alpha) & \text{if } a=0, b=1+\beta, c=0 \\
(d) & \delta = (2n+1+\alpha)(2n+2+\gamma-\beta) & \text{if } a=1+\alpha, b=0, c=0 \\
(e) & \delta = (2n+2+\alpha+\beta)(2n+3+\gamma) & \text{if } a=1+\alpha, b=1+\beta, c=0 \\
(f) & \delta = (2n+2-\gamma+\beta)(2n+3-\alpha) & \text{if } a=0, b=1+\beta, c=1-\gamma \\
(g) & \delta = (2n+2-\gamma+\alpha)(2n+3-\beta) & \text{if } a=1+\alpha, b=0, c=1-\gamma \\
(h) & \delta = (2n+4)(2n+3-\gamma+\alpha+\beta) & \text{if } a=1+\alpha, b=1+\beta, c=1-\gamma
\end{array} \tag{18}$$

### *3.2. Multiple algebraization*

For some values of $\alpha, \beta, \gamma$, two or more of the families (18) are identical. The associated Heun equation then possesses several types of algebraic solutions; we will say that it has a *multiple algebraization*. The Lamé equation [8]

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} + (\lambda - N(N+1)k^2\operatorname{sn}^2)F = 0 \tag{19}$$

provides a typical example of such a situation. It corresponds to $\alpha = \beta = \gamma = 0$ in equation (3). A total number of $2N+1$ algebraic solutions exist; they are known as Lamé polynomials. They are of the types $(a)$, $(e)$, $(f)$, $(g)$ (respectively, of types $(b)$, $(c)$, $(d)$, $(h)$) if $N$ is even (respectively, odd) [9]. The same pattern of the algebraic solutions occurs, in fact, as long as $\alpha, \beta, \gamma$ are even integers.

As another example, let $\alpha = \frac{1}{2}$, $\beta = \gamma = 0$. The families $(a)$ and $(f)$ of equation (18) in equation (14) coincide, so that algebraic solutions of the corresponding types coexist. The same statement holds true for families $(b)$ and $(c)$, $(d)$ and $(h)$ and, finally, for $(e)$ and $(g)$.

For odd values of $\alpha$ and (or) $\beta$ and (or) $\gamma$, the degeneracies in $a$ and (or) $b$ and (or) $c$ predicted by equation (15) are only apparent. Indeed, the corresponding factor(s) in equation (7) is (are) polynomial(s) in $t$ and, accordingly, is (are) absorbed into $P_n$.

### 3.3. The case $\beta = \gamma = 0$

Let us now concentrate on the case $\beta = \gamma = 0$. Operator (3) is then regular everywhere. Moreover, the limit $k = 0$ corresponds to the free Schrödinger equation on the circle ($K(0) = \pi/2$). The corresponding spectrum is well known: $\lambda = 0, 1, 1, 4, 4, \ldots j^2, j^2, \ldots$. Using obvious notation, we label the $k \neq 0$ eigenvalues emerging from these values at $k = 0$ according to $S_0, S_1, A_1, S_4, A_4, \ldots$. The explicit solutions corresponding to $n = 0$ and $n = 1$ in equation (18) are given in appendix A.

In the case of Lamé equation (19), the $2N + 1$ Lamé polynomials can be recovered from appendix A (setting $\alpha = 0$) for $N = 0, 1, 2, 3$. Inspection of the limit $k = 0$ suggests that the Lamé polynomials occupy the $2N + 1$ lowest energy levels.

For $\alpha = 2$, the possible values of $\delta$ are also of the form $\delta = N(N+1)$ ($N$ integer). The algebraic solutions corresponding to the lowest values of $N$ occupy the following levels:

$$\delta = 0 \qquad S_0(a_0), A_1(b_0), S_1(c_0) \tag{20}$$

$$\delta = 2 \qquad S_0(a_1), A_4(f_0), S_4(a_1) \tag{21}$$

$$\delta = 6 \qquad S_0(d_0), A_1(b_1), S_1(c_1), A_9(b_1), S_9(c_1) \tag{22}$$

where we have labelled the eigenstates according to their value in the $k = 0$ limit (the labels in parentheses $a_n, b_n, \ldots$, with $n = 0$ or 1 refer to appendix A). Similarly, for $\alpha = \frac{1}{2}$, one finds

$$\delta = \tfrac{1}{2} \qquad A_1(b_0), S_1(c_0) \tag{23}$$

$$\delta = 3 \qquad S_0(d_0) \tag{24}$$

$$\delta = 5 \qquad S_0(a_1), A_4(f_1), S_4(a_1). \tag{25}$$

These cases show explicitly that, in general, non-algebraic eigenstates occur between the algebraic eigenstates (in contrast to the Lamé equation). For the higher values of $\delta$, the set of the algebraic solutions involve higher values of $n$ than those given in the appendix.

For $\beta = \gamma = 0$, the potential (7) is regular and bounded. The eigenstates of equation (6) are given by

$$\psi = \frac{F}{\mathrm{dn}^{\alpha/2}} \qquad h = \lambda - \tfrac{1}{4}(\alpha^2 + 2\alpha - k^2\alpha^2). \tag{26}$$

### 3.4. More solutions from the trigonometric form

Expanding the eigenfunctions of operator $\tilde{H}$ (see equation (12)) according to

$$\Phi(\phi) = f_0(z)E_0 + f_1(z)E_1 + f_2(z)E_2 + f_3(z)E_3 \qquad z \equiv \sin^2\phi \tag{27}$$

$$E_0 = 1 \qquad E_1 = \cos\phi \qquad E_2 = \sin\phi \qquad E_3 = \cos\phi\sin\phi \tag{28}$$

enables one to transform equation (11) into four decoupled equations (one for each function $f_a$, $a = 0, 1, 2, 3$). All algebraic solutions mentioned above can be recovered in this way.

One may wonder, however, whether alternative ansatzes to equation (27) can be used for $\Phi(\phi)$ which lead to different types of algebraic solutions. In particular, it would be interesting to obtain, in this way, the doubly-periodic solutions of the Lamé equation [8]. One possibility consists in choosing $\Phi(\phi)$ of the form

$$\Phi(\phi) = h_1(z)\left(\sin\frac{2\phi-\pi}{4}\right)^{1-\gamma} + h_2(z)\left(\sin\frac{2\phi-\pi}{4}\right)^{1-\gamma}\sin\phi \qquad z \equiv \sin^2\phi. \tag{29}$$

It allows one to transform equation (11) into a system of two coupled equations for $h_1(z), h_2(z)$. The new equations have polynomial coefficients of $z$, provided $\beta = 0$. If, in addition, $\alpha = 0$, then the resulting $2 \times 2$ matrix operator preserves the following spaces:

$$\mathcal{P}(p,p) \qquad \text{if} \qquad \eta = p + \frac{3-\gamma}{4} \qquad p \text{ integer} \tag{30}$$

$$\mathcal{P}(p,p-1) \qquad \text{if} \qquad \eta = p + \frac{1-\gamma}{4} \qquad p \text{ integer.} \tag{31}$$

Here, $\mathcal{P}(m,n)$ denotes the vector space of doublets of polynomials of degrees $m$ and $n$. In this way, algebraic solutions are obtained which are doubly periodic, i.e. with period $4\pi$. To each of these, one can associate another algebraic solution (with the same eigenvalue) by shifting the variable $\phi$ by a quarter of a period (i.e. by $\pi$). This is equivalent to replacing (20) by

$$F(\phi) = h_1(z)\left(\cos\frac{2\phi-\pi}{4}\right)^{1-\gamma} + h_2(z)\left(\cos\frac{2\phi-\pi}{4}\right)^{1-\gamma}\sin\phi. \tag{32}$$

The values of $\delta$ associated with these solutions differ from those of the form (18); ansätze (29), (32) really lead to different algebraic solutions.

We have failed to obtain solutions of this type for general values of $\alpha, \beta, \gamma$. Doubly periodic solutions were also obtained for $\alpha = \gamma = 0$; the counterpart of ansatz (29) then reads

$$\Phi(\phi) = h_1(z)\left(\sin\frac{\phi}{2}\right)^{1+\beta} + h_2(z)\left(\sin\frac{\phi}{2}\right)^{1+\beta}\cos\phi \tag{33}$$

$$H = h_1(z)\left(\cos\frac{\phi}{2}\right)^{1+\beta} + h_2(z)\left(\cos\frac{\phi}{2}\right)^{1+\beta}\cos\phi. \tag{34}$$

## 4. The Heun equation and sphaleron stability

There are several examples of fields theories (especially in $1+1$ dimensions, e.g. [10–12]) which possess non-trivial classical solutions; the best known of these solutions are the solitons and sphalerons [13]. They correspond, respectively, to local minima and unstable extrema of the classical energy. In a few cases, the classical solution can be computed in a closed form and the stability analysis can be formulated in terms of a Heun equation [10, 14–16].

The Abelian–Higgs model in $1+1$ dimensions admits a sphaleron solution [17]. The stability of the solution can be carried out by studying the fluctuations about the sphaleron. With a suitable parametrization of the fluctuations, the normal-mode equation reads [18, 19]

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} \theta^2 k^2 \,\mathrm{sn}^2 & 2\theta k \,\mathrm{cn}\,\mathrm{dn} \\ 2\theta k \,\mathrm{cn}\,\mathrm{dn} & (2+\theta^2)k^2\,\mathrm{sn}^2 - 1 - k^2 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix} \tag{35}$$

where $\theta^2$ represents the mass ratio of the Higgs boson to the gauge boson. If $\theta^2 = N(N+1)$, where $N$ is an integer, then $4N+2$ solutions of system (35) can be obtained by solving algebraic equations [19, 20]. For instance, in the cases $N=1$ and $N=2$, the state of the lowest eigenvalue reads, respectively,

$$f = \sqrt{2}k\,\mathrm{cn}\,\mathrm{dn} \qquad g = 2k^2\,\mathrm{sn}^2 - \lambda \qquad \lambda = -(1+k^2) \tag{36}$$

$$\begin{aligned} &f = \sqrt{6}k\,\mathrm{cn}(6k^2\,\mathrm{sn}^2 + \lambda - 3) \qquad g = -\mathrm{dn}(18k^2\,\mathrm{sn}^2 + \lambda - 3 - 12k^2) \\ &\lambda = 1 - 2\sqrt{1+3k^2}. \end{aligned} \tag{37}$$

The eigenvalue $\lambda$ is a negative function of $k$. Accordingly, the eigenmode above represents the direction of instability of the sphaleron.

The occurence of a finite number of algebraic solutions to equation (35) suggests that it might be related to the Heun equations. One can further pursue the analogy by expressing spectral problem (35) in terms of new functions, say $F(x)$, $W(x)$, defined by [24]

$$F(x) = f(x) \qquad W(x) = \frac{\mathrm{d}f}{\mathrm{d}x} - \theta k\,\mathrm{sn}(x,k)g(x). \tag{38}$$

With this reparametrization, equation (35) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}W + (\lambda - \theta^2k^2\,\mathrm{sn}^2)W = 0 \tag{39}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F - \frac{2\,\mathrm{cn}\,\mathrm{dn}}{\mathrm{sn}}\frac{\mathrm{d}F}{\mathrm{d}x} + (\lambda - \theta^2k^2\,\mathrm{sn}^2)F = \frac{-2\,\mathrm{cn}\,\mathrm{dn}}{\mathrm{sn}}W. \tag{40}$$

Equation (39) decouples and is nothing but the Lamé equation. The differential operator on the left-hand side of equation (40) is of the form (4) with $\alpha = \beta = 0$, $\gamma = -2$. As shown in the previous section, this operator admits $2N+1$ algebraic eigenvectors. Therefore $2N+1$ solutions of equation (35), that is to say half of the algebraic normal modes (in particular, the negative mode (36),(37)), correspond to $W=0$ and are determined by a Heun equation. The other algebraic solutions correspond to the $2N+1$ Lamé polynomials of equation (39).

For the first few values of $N$, the algebraic solutions of equation (40) (with $W=0$) read:

$$N=1 \qquad \begin{aligned} &F_0 = \mathrm{cn}\,\mathrm{dn} \qquad \lambda_0 = -(1+k^2) \\ &F_{1,2} = k\,\mathrm{sn}^2 \pm 1 \qquad \lambda_{1,2} = \pm 2k \end{aligned} \tag{41}$$

$$N=2 \qquad \begin{aligned} &F_{0,3} = \mathrm{cn}(6k^2\,\mathrm{sn}^2 + \lambda_{0,3} - 3) \qquad \lambda_{0,3} = 1 \mp 2\sqrt{1+3k^2} \\ &F_{1,2} = \mathrm{dn}(6k^2\,\mathrm{sn}^2 + \lambda_{1,2} - 3k^2) \qquad \lambda_1 = k^2 \mp 2k\sqrt{3+h^2} \\ &F_4 = \mathrm{sn}^3 \qquad \lambda_4 = 3(1+k^2). \end{aligned} \tag{42}$$

This example suggests that QES operators can be interesting on their own, even though the associated Schrödinger equation is not. The Schrödinger equation associated with equation (40) (with $W=0$), through equation (4), reads:

$$-\psi'' + \left(2k^2\,\mathrm{sn}^2 + \frac{2}{\mathrm{sn}^2}\right)\psi = h\psi \qquad h = (\lambda + 1 + k^2) \qquad \psi = \frac{F}{\mathrm{sn}}. \tag{43}$$

The potential here is singular and so are the algebraic eigenfunctions.

## 5. Supersymmetry

Starting from a Schrödinger operator, say $H_-$, shifted in such a way that the energy of the ground state, say $\psi_0$, vanishes, the following relations hold:

$$H_- \equiv \left(\frac{\mathrm{d}}{\mathrm{d}x} + W\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W\right) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}W}{\mathrm{d}x} + W^2 \tag{44}$$

$$W = \frac{1}{\psi_0}\frac{\mathrm{d}\psi_0}{\mathrm{d}x} \qquad -\frac{\mathrm{d}}{\mathrm{d}x}\psi_0 + W\psi_0 = 0. \tag{45}$$

It is well known [21] that the spectrum of the operator

$$H_+ \equiv \left(-\frac{\mathrm{d}}{\mathrm{d}x} + W\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + W\right) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}W}{\mathrm{d}x}\right) + W^2 \tag{46}$$

is related to the spectrum of $H_-$. Indeed, if $H_-\psi = h\psi$, then

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W\right)H_-\psi = H_+\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W\right)\psi = h\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W\right)\psi. \tag{47}$$

Hence, the spectrum of $H_+$ is the same as the spectrum of $H_-$, apart from the state of lowest energy which is annihilated by the operator $((-\mathrm{d}_x + W)\psi_0 = 0)$. However, since we have

$$H_+(\psi_0^{-1}) = 0 \tag{48}$$

it appears that if, by chance, $\psi_0^{-1}$ is a normalizable function (as may be the case for periodic potentials), then the two Hamiltonians $H_+$ and $H_-$ possess exactly the same spectrum. The construction of supersymmetric partners of physically relevant Lamé equations is discussed in [15]. The constuction of supersymmetric partners of QES operators was investigated in [22, 23]; here we want to discuss it more specifically for Heun equations. Obviously, the branches labelled $(a)$ and $(d)$ in appendix A correspond to ground states (the eigenvalue is zero for $k^2 = 0$). For $n = 0$, the superpotentials associated with branches $(a)$ and $(d)$ read, respectively,

$$W = k^2\frac{\alpha}{2}\frac{\mathrm{sn}\,\mathrm{cn}}{\mathrm{dn}} \qquad W = -k^2\frac{\alpha+2}{2}\frac{\mathrm{sn}\,\mathrm{cn}}{\mathrm{dn}} \tag{49}$$

and lead, for $H_+$, to potentials of the form (7). The superpotentials associated with branches $(a)$ and $(d)$ for $n = 1$, read, respectively,

$$W = \frac{\mathrm{sn}\,\mathrm{cn}}{2\,\mathrm{dn}}\left(\frac{4 - 2k^2(\alpha/\lambda) + k^2(\alpha-4)\,\mathrm{sn}^2}{\mathrm{sn}^2 - (2/\lambda)}\right)$$
$$\lambda = 2 + (2-\alpha)k^2 - \sqrt{k^4(\alpha-2)^2 + 4 - 4k^2} \tag{50}$$

$$W = \frac{\mathrm{sn}\,\mathrm{cn}}{2\,\mathrm{dn}}\left(\frac{4 + 2k^2(\alpha+2)/\rho - k^2(\alpha+6)\,\mathrm{sn}^2}{\mathrm{sn}^2 - (2/\rho)}\right)$$
$$\rho = 2 + (4+\alpha)k^2 - \sqrt{k^4(\alpha+4)^2 + 4 - 4k^2} \tag{51}$$

and lead, for $H_+$, to more complicated potentials than (7).

The authors of [15] considered the sphaleron solutions available in the sine–Gordon model and in the double-well potential considered in 1+1 dimensions with periodic boundary condition of the space variable. They pointed out that the normal-mode equation about the sphaleron reduces to Lamé equation (19) with $N = 1$ and $N = 2$, respectively. They also focused their attention on the supersymmetric partners of these equations, obtaining, for the superpotentials, expressions which are particular cases of our equations (50) and (51). Owing to the role of system (33), we attempted to construct the analogue of the superpotential for systems of coupled Schrödinger equations.

Consider a generic $2\times 2$-matrix Schrödinger operator $-\frac{d^2}{dx^2}+V(x)$ and let $\psi = (f_0, g_0)$ denote its ground state (assuming that the ground-state energy is zero). Then, define the vectors

$$\hat{\psi} = \frac{1}{p}\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \qquad \hat{\chi} = \frac{1}{p}\begin{pmatrix} -g_0 \\ f_0 \end{pmatrix} \qquad p(x) = \sqrt{(f_0^2 + g_0^2)} \tag{52}$$

which are normalized and orthogonal at any space point $x$, so that a scalar function $a(x)$ exists such that

$$\frac{d}{dx}\hat{\psi} = a(x)\hat{\chi} \qquad \frac{d}{dx}\hat{\chi} = -a(x)\hat{\psi}. \tag{53}$$

It appears that the superpotential $W(x)$, such that $V = W' + W^2$, can be written in the form

$$W(x) = \frac{1}{p}\frac{dp}{dx}|\hat{\psi}\rangle\langle\hat{\psi}| + a(x)(|\hat{\psi}\rangle\langle\hat{\chi}| + |\hat{\chi}\rangle\langle\hat{\psi}|) + b(x)|\hat{\chi}\rangle\langle\hat{\chi}| \tag{54}$$

(using Dirac notation) provided that function $b(x)$ obeys the Ricatti equation

$$\frac{db}{dx} + b^2 = \langle\hat{\chi}|V|\hat{\chi}\rangle - 3a^2. \tag{55}$$

This equation specifies how to constuct, in principle, the supersymmetric partner of the $2\times 2$ matrix potential $V(x)$. Unfortunately, we failed to obtain explicit solutions for $b(x)$ when the ground state $(f_0, g_0)$ is given by equation (36) or (37).

## 6. Conclusions

We have reconsidered the Heun equation from the point of view of the notion of quasi-exact solvability. In view of possible applications, we have put the Heun equation in several different, but equivalent, forms.

We believe that suitable ansatzes for the eigenfunction can be performed that transform Heun equations into QES systems. This procedure makes the notion of quasi-exact solvability more flexible and enlarges the set of 'algebraically' accessible solutions. Although we succeeded with this programme for particular values of the parameters, we hope that the technique could be adapted for more general cases.

System (35) has many of the properties of Heun equations; namely, it possesses a multiple algebraization. The question therefore arises whether we can classify the systems of two (or more) Heun-like equations and their algebraic solutions. This is an interesting mathematical problem; moreover, it would be profitable to be able to treat such systems systematically in view of some other physical applications, e.g. those issued from low-dimensional field theories and treated as a testing ground for more realistic applications.

## Appendix A

In this appendix, we give the solutions of equation (3) for $\beta = \gamma = 0$, $\delta$ of the form (18) and $n = 0$ and $n = 1$. Higher values of $n$ need to factorize polynomials of degree higher than two.

### A.1. Solutions for n=0

| | | | |
|---|---|---|---|
| $(a)$ | $\delta = 0$ | $F = 1$ | $\lambda = 0$ |
| $(b)$ | $\delta = 2 - \alpha$ | $F = \mathrm{sn}$ | $\lambda = 1 + k^2(1-\alpha)$ |
| $(c)$ | $\delta = 2 - \alpha$ | $F = \mathrm{cn}$ | $\lambda = 1$ |
| $(d)$ | $\delta = 2 + 2\alpha$ | $F = \mathrm{dn}^{1+\alpha}$ | $\lambda = k^2(1+\alpha)$ |
| $(e)$ | $\delta = 6 + 3\alpha$ | $F = \mathrm{cn}\,\mathrm{dn}^{1+\alpha}$ | $\lambda = 1 + k^2(1+\alpha)$ |
| $(f)$ | $\delta = 6 - 2\alpha$ | $F = \mathrm{cn}\,\mathrm{sn}$ | $\lambda = 4 + k^2(1-\alpha)$ |
| $(g)$ | $\delta = 6 + 3\alpha$ | $F = \mathrm{sn}\,\mathrm{dn}^{1+\alpha}$ | $\lambda = 1 + k^2(4+2\alpha)$ |
| $(h)$ | $\delta = 12 + 4\alpha$ | $F = \mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}^{1+\alpha}$ | $\lambda = 4 + k^2(4+2\alpha)$ |

### A.2. Solutions for n=1

$(a)$ $\delta = 6 - 2\alpha$

$$F = \mathrm{sn}^2 - \frac{2}{\lambda_\pm}$$

$$\lambda_\pm = 2 + (2-\alpha)k^2 \pm \sqrt{k^4(\alpha-2)^2 - 4k^2 + 4}$$

$(b)$ $\delta = 12 - 3\alpha$

$$F = \mathrm{sn}\left(\mathrm{sn}^2 - \frac{6}{k^2(\alpha-1) + \lambda_\pm - 1}\right)$$

$$\lambda_\pm = 5 + (5-2\alpha)k^2 \pm \sqrt{k^4(\alpha-4)^2 + 4k^2(\alpha-7) + 16}$$

$(c)$ $\delta = 12 - 3\alpha$

$$F = \mathrm{cn}\left(\mathrm{sn}^2 - \frac{2}{\lambda_\pm - 1}\right)$$

$$\lambda_\pm = 5 + (2-\alpha)k^2 \pm \sqrt{k^4(\alpha-2)^2 - 4k^2(1+\alpha) + 16}$$

$(d)$ $\delta = 12 + 4\alpha$

$$F = \mathrm{dn}^{1+\alpha}\left(\mathrm{sn}^2 + \frac{2}{k^2(\alpha+1) - \lambda_\pm}\right)$$

$$\lambda_\pm = 2 + (5+2\alpha)k^2 \pm \sqrt{k^4(\alpha+4)^2 - 4k^2 + 4}$$

$(e)$ $\delta = 5(4+\alpha)$

$$F = \mathrm{cn}\,\mathrm{dn}^{1+\alpha}\left(\mathrm{sn}^2 + \frac{2}{k^2(1+\alpha) + 1 + \lambda_\pm}\right)$$

$$\lambda_\pm = 5 + (5+2\alpha)k^2 \pm \sqrt{k^4(\alpha+4)^2 + 4k^2(\alpha+1) + 16}$$

$(f)$ $\delta = 4(5-\alpha)$

$$F = \mathrm{cn}\,\mathrm{sn}\left(\mathrm{sn}^2 - \frac{6}{k^2(\alpha-1) + \lambda_\pm - 4}\right)$$

$$\lambda_\pm = 10 + (5-2\alpha)k^2 \pm \sqrt{k^4(\alpha-4)^2 - 36k^2 + 36)}$$

$$(g) \qquad \delta = 5(4+\alpha) \qquad F = \operatorname{cn}\operatorname{dn}^{1+\alpha}\left(\operatorname{sn}^2 + \frac{6}{2k^2(\alpha+2) - \lambda_\pm + 1}\right)$$

$$\lambda_\pm = 5 + (10+3\alpha)k^2 \pm \sqrt{k^4(\alpha+6)^2 - 4k^2(9+\alpha) + 16}$$

$$(h) \qquad \delta = 6(5+\alpha) \qquad F = \operatorname{sn}\operatorname{cn}\operatorname{dn}^{1+\alpha}\left(\operatorname{sn}^2 + \frac{6}{2k^2(\alpha+2) + 4 - \lambda_\pm}\right)$$

$$\lambda_\pm = 10 + (10+3\alpha)k^2 \pm \sqrt{k^4(\alpha+6)^2 - 36k^2 + 36}$$

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